Scaling limit of the one-dimensional XXZ Heisenberg chain with easy axis anisotropy

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Abstract

We construct the scaling limit of the easy axis XXZ chain. This limit is a subtle combination of approaching the isotropic point, and letting the lattice spacing to zero to obtain a continuous model with a finite mass gap. We give the energy difference between the two lowest energy states (the two 'vacua') and analyze the structure of the excitation spectrum of the limiting model. We find, that the excitations form two sets corresponding to the two vacua. In both sets the dressed particles are described by Bethe Ansatz like equations (higher level Bethe Ansatz), and the two sets can be distinguished through a parameter entering into these secular equations. The degenerations in the spectrum can be interpreted as originating from an SU(2) symmetry of the dressed particles. The two particle scattering matrices obtained from the secular equations are consistent with this symmetry, and they differ in an overall sign in the two sectors.

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I. INTRODUCTION

The XXZ Heisenberg chain is defined by the Hamiltonian

$$H_{XXZ} = \frac{V}{a} \sum_{n=1}^{N} \left(\left(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y \right) + \rho \left(S_n^z S_{n+1}^z - 1/4 \right) \right) . \tag{1.1}$$

The Hilbert space is the tensor product of N spaces furnishing the doublet representation of SU(2). $S_n^{x,y,z}$ are the spin operators acting on the nth site and in (1.1) the $(N+1)$th site is identified with the first one. We consider the easy axis region defined by

$$\rho = \cosh \gamma > 1 . \tag{1.2}$$

The V/a factor with a being the lattice spacing and $V = 2/\pi$ is included for the sake of proper normalization.

The Hamiltonian (1.1) can be diagonalized by Bethe Ansatz (BA) [1–4], the BA equations (BAE) were analyzed in [5–7]. Due to these studies it is well known by now, that the chain with even number of sites has two states in which there are no free parameters [6]. These are the ground state and the first excited state, and as the difference in their energy disappears exponentially fast with the length of the chain N [7,8], they are often referred to as the two ground states, although it is more precise to call them the two vacua. The other excited states are described in terms of a few parameters characterizing some kind of dressed particles. These posses a gap, and their parameters satisfy BA type equations, the so called higher level Bethe Ansatz equations (HLBAE) [5,6]. The excited states form two groups which can be associated with the two ground states [6].

The model has two relativistic continuum limits [9]. One is constructed by setting $\gamma = 0$ first and taking the $a \to 0$ (together with $N \to \infty$ but $Na = L$) continuum limit afterwards resulting a massless SU(2) conformal field theory. To obtain the other relativistic limit one has to perform the $\gamma \to 0$ and $a \to 0$ simultaneously keeping the gap in the properly normalized spectrum constant. The aim of the present work is to study the details of this second limit, and analyze the limiting theory. Our results are as follows.

- The limiting theory is a massive relativistic theory with a mass M, if the continuum limit is performed in such a way, that

$$\gamma \to 0 , \qquad a \to 0 , \qquad \frac{4}{a} \exp \left\{ -\frac{\pi^2}{2\gamma} \right\} \to M . \tag{1.3}$$

  As the length of the chain $L = Na$ is kept finite, also the number of sites must be adjusted:

$$N = \frac{LM}{4} \exp\left\{\frac{\pi^2}{2\gamma}\right\} \to \infty \,. \tag{1.4}$$

Our expression for the mass M differs from that found in [9] in a prefactor of the exponential.

- In the above limit the difference in the energies of the two 'ground states' (physical vacua) stays finite but exponentially small in the length of the system:

$$\Delta E_0 = \sqrt{\frac{8M}{\pi L}} e^{-LM} \,. \tag{1.5}$$

- The excited states of the system can be described in terms of excitations (dressed particles). Each particle is characterized by a rapidity we denote by ϑ. The energy and momentum contribution of the particles are the sums of the contributions of the individual particles

$$E - E_0 = \sum \varepsilon(\vartheta)\,, \qquad P = \sum p(\vartheta)\,, \tag{1.6}$$

with

$$\varepsilon(\vartheta) = M \cosh\vartheta\,, \qquad p(\vartheta) = M \sinh\vartheta\,. \tag{1.7}$$

The rapidities of the particles must satisfy a set of BA type equations:

$$Lp(\vartheta_h) = 2\pi(\mathcal{I}_h + I_0) - \sum_l^{n(\vartheta)} \phi\left(\frac{\vartheta_h - \vartheta_l}{\pi}\right) + \sum_\alpha^{n(\kappa)} 2\tan^{-1}\frac{\vartheta_h - \kappa_\alpha}{\pi/2}\,,$$

$$\mathcal{I}_h = \frac{1}{2}\left(\frac{n(\vartheta) - 2n(\kappa)}{2}\right) \text{(od 1)}\,, \qquad I_0 = 0 \text{ or } 1/2\,, \tag{1.8a}$$

and

$$\sum_h^{n(\vartheta)} 2\tan^{-1}\frac{\kappa_\alpha - \vartheta_h}{\pi/2} = 2\pi\mathcal{J}_\alpha + \sum_\beta^{n(\kappa)} 2\tan^{-1}\frac{\kappa_\alpha - \kappa_\beta}{\pi}\,,$$

$$\mathcal{J}_\alpha = \left(\frac{n(\kappa) - n(\vartheta) + 1}{2}\right) \text{(od 1)}\,. \tag{1.8b}$$

Here $n(\vartheta)$ is the number of particles, the set of variables κ is needed to describe the internal symmetry of the states, their number $n(\kappa)$ obeys $n(\vartheta) - 2n(\kappa) \geq 0$, and

$$\phi(x) = \frac{1}{i}\ln\frac{\Gamma\left(\frac{1}{2} - i\frac{x}{2}\right)\Gamma\left(1 + i\frac{x}{2}\right)}{\Gamma\left(\frac{1}{2} + i\frac{x}{2}\right)\Gamma\left(1 - i\frac{x}{2}\right)}\,. \tag{1.9}$$

- The excited states form SU(2) multiplets in which the energy is the same. Each *multiplet* is characterized by one solution of the equations (1.8). The number of states belonging to one multiplet is $(n(\vartheta) - 2n(\kappa)) + 1$. Within a multiplet the states are labeled by the value of the S^z taking the values $n(\vartheta)/2 - n(\kappa)$, $n(\vartheta)/2 - n(\kappa) - 1$, ..., $-n(\vartheta)/2 + n(\kappa)$. This indicates, that the particles have SU(2) symmetry or a symmetry producing the same multiplet structure, and the z component of the spin connected to this symmetry coincides with the z component of the real spins building up the original chain.

- The solutions of (1.8) form two groups, one for $I_0 = 0$ and one for $I_0 = 1/2$. We argue, that the two sets of excited states are the excitations of the two vacua. In both sets the $S^z = 0$ states, including the vacua, are eigenstates of flipping all spins, but with different eigenvalues. In the case of the two particle excitations the symmetry of the singlets is the same as that of the corresponding vacuum, and that of the triplets is the opposite. This is the same structure as found in the XXX chain, where it is a consequence of the SU(2) symmetry of the Hamiltonian (and this structure can be interpreted in terms of SU(2) with a modified coproduct, or equivalently in terms of a q-deformed SU(2) at $q = -1$).

- The two particle scattering matrix can be given up to an overall phase as

$$\hat{S}(\Delta\vartheta) = -\exp\left\{i\left(2\pi I_0 + \phi\left(\frac{\Delta\vartheta}{\pi}\right)\right)\right\}\left(\hat{P}_{tr} + \frac{\Delta\vartheta + i\pi}{\Delta\vartheta - i\pi}\hat{P}_s\right), \qquad (1.10)$$

  where $\hat{P}_{tr}$ and $\hat{P}_s$ are the projectors on the triplet resp. singlet subspace of the two spins.

All these shows, that the scaling limit (SL) of the XXZ yields a massive relativistic theory (in duplicate) which is of the same structure as the massive sector of the theory obtained through the SL of the attractive Hubbard chain and identified as a regularization of the SU(2) symmetric chiral invariant Gross-Neveu (CGN) model [10,11]. It is widely accepted, that the XYZ chain is a lattice regularization of the sine-Gordon (SG) theory or the massive Thirring model (MTM) [12] (as these latter two are equivalent [13]). It is also known, that the MTM/SG theory at a special value of the coupling is equivalent to the CGN model (up to a free massless boson field), more over this value of the coupling corresponds to the antiferromagnetic (our $\rho=1$) point of the XYZ chain. This way recovering the massive sector of the CGN model is not unexpected. It is remarkable, however, that the SL of the XXZ chain is taken through couplings not corresponding to stable MTM/SG theories. Our detailed analysis also rises some questions connected with the mass-formula and the symmetry of the vacua.

The (1.3) mass-formula is not of the form found in the case of the Hubbard chain (a prefactor $\sqrt{\gamma}$ is 'missing'), and it defines a β-function different from the one expected based on perturbation calculations. This may be connected with the fact, that the $\rho=1$ point in the

space of the couplings of the XYZ model is singular (in the sense, that certain quantities like the ground-state energy of the XXZ chain are singular in this point [3]), thus approaching this point from different directions may lead to different results. In our SL the isotropic point is approached along the XXZ line while the limit taken in [12] involves the XY-type anisotropy.

Another interesting point is the existence of two very similar but not completely identical limiting theories. The two theories are distinguished by the symmetry of the vacuum with respect to reversing all spins. One has to note, however, that measuring this symmetry in the continuum limit may encounter difficulties, as the SL of the corresponding operator can not be constructed directly. This symmetry has not been studied in other representations of the CGN model although it may be present also in other cases. In the SL of the half-filled Hubbard chain it is even stronger in the sense that the vacuum is a singlet of both the spins and the isospins, and it is symmetric under both the spin and isospin reversal.

The paper is organized as follows. In Section II we summarize those properties of the XXZ chain we need to construct its scaling limit and we also propose a formula to calculate the symmetry with respect to flipping all spins. In Section III the SL is constructed and the properties of the limiting theory are analyzed. Specially we construct the limiting process resulting in a relativistic dispersion (III A), analyze the vacua in this limit (III B), derive the secular equations of the limiting theory (III C), we give the SU(2) multiplet structure of the eigenstates (III D), derive the two particle S-matrix (III E), we study the reflection symmetry of the two particle $S^z = 0$ states (III F), and we also argue, that the boundary condition obeyed by the particles of the limiting theory is connected to the parity properties of N kept unchanged in the $N \to \infty$ limit (III G). The more technical details are collected in appendices. The properties of the elliptic functions we use are listed in Appendix A. The behavior of the two vacua are calculated in Appendix B, and in Appendix C we check if the the approximations used to derive the HLBAE of the lattice model remain valid also in the SL. In Appendix D we describe the structure of the solutions of the HLBAE and give the classes of solutions which become degenerate in the SL. Finally some details of our numerical calculations are given in Appendix E.

II. BETHE ANSATZ SOLUTION OF THE XXZ CHAIN

A. The BA equations

This section is intended to summarize the BA solution of the XXZ chain [5–7] with emphasis on those points which are relevant to the construction of the SL.

First let us recall some properties of the Hamiltonian (1.1). In the literature two versions of the XXZ model are commonly found which are distinguished by the relative sign of the

$(S_n^x S_{n+1}^x + S_n^y S_{n+1}^y)$ and $(S_n^z S_{n+1}^z)$ terms. The two theories are equivalent if N is even, the corresponding transformation on the Hilbert spaces is generated by $\prod_k \sigma_{2k}^z$ which relates the spin operators as: $\{S_{2k}^x, S_{2k}^y, S_{2k}^z\} \to \{-S_{2k}^x, -S_{2k}^y, S_{2k}^z\}$. As this is a symmetry of the Hamiltonian, the energy spectrum does not change, but it does have an effect on the eigenvalues of the momentum operator which is relevant to the connection to the continuum limit. The momentum of every spin wave is shifted by π which affects all the states with an odd number of spin waves, including the ground state whenever $N/2$ is an odd integer. There is another useful operator, which commutes with the Hamiltonian, but whose eigenvalues are affected:

$$\hat{\Sigma} = \prod_{n=1}^{N} \sigma_n^x \,, \tag{2.1}$$

which represents a reflection on the x-axis. (In a basis given by the products of the S^z eigenstates of the individual spins $\hat{\Sigma}$ simply flips all the spins (the up ones down and the down ones up)). The symmetry connected with this operation we call spin reversal or reflection symmetry.) The $S^z = 0$ BA eigenstates of (1.1) are expected to be also eigenstates of (2.1). For the sake of definiteness we have chosen the positive sign in (1.1) and note that when N is an integer multiple of four, the results should agree with the ones obtained from the opposite convention.

The Hamiltonian (1.1) and S^z can be simultaneously diagonalized by the Bethe Ansatz (BA) and due to the symmetry corresponding to inverting the spins it is sufficient to consider only $S^z \geq 0$ states. The eigenvectors are explicitly given in terms of a wave number set $|\{k_1 \ldots k_r\}\rangle$, for the states with $S^z = N/2 - r$, $(r \leq N/2)$

$$|\{k_1 \ldots k_r\}\rangle = \sum_{n_1 < \ldots < n_r} \left(\sum_{\mathcal{P}} \exp\left\{ i \sum_\alpha k_{\mathcal{P}_\alpha} n_\alpha + \frac{i}{2} \sum_{\alpha < \beta} \Psi_{\mathcal{P}_\alpha \mathcal{P}_\beta} \right\} \prod_{n_\alpha} \sigma_{n_\alpha}^- |F\rangle \right), \tag{2.2}$$

where the summation is over all permutations $\mathcal{P}$ of $\{1, \ldots, r\}$ and the wave numbers k_α together with the phase shifts $\Psi_{\alpha\beta}$ satisfy the equations

$$\cot \frac{\Psi_{\alpha,\beta}}{2} = -\rho \frac{\cot \frac{k_\alpha}{2} - \cot \frac{k_\beta}{2}}{(1-\rho)\cot \frac{k_\alpha}{2} \cot \frac{k_\beta}{2} - (1+\rho)} \,, \tag{2.3}$$

$$\exp\{N k_\alpha\} = \exp\left\{ \sum_{\beta(\neq\alpha)} \Psi_{\alpha,\beta} \right\}. \tag{2.4}$$

The energy and the momentum of the state are

$$E = \frac{V}{a} \sum_\alpha (\cos k_\alpha - \rho) \,; \qquad Q = \sum_\alpha k_\alpha \,. \tag{2.5}$$

The equation (2.3) can be solved for the phase shifts by parameterizing k_α's in terms of new variables, v_α [4] which are commonly called rapidities:

$$k_\alpha = \Phi(v_\alpha, \gamma/2), \qquad \Psi_{\alpha,\beta} = \Phi(v_\alpha - v_\beta, \gamma), \tag{2.6}$$

with

$$\Phi(z, \delta) \equiv \frac{1}{i} \ln \frac{\sin(z + i\delta)}{\sin(z - i\delta)}. \tag{2.7}$$

When expressing (2.4) in terms of the rapidities it turns out to be useful to take the logarithm of the equation. In order to do this one has to define the branch cuts of $\Phi(z, \delta)$. We choose $\Phi(z, \delta)$ to be a continuous function of z with $\Phi(0, \delta) = \pi$ in the strip $|\text{Im } z| < \delta$, while in the regions $\text{Im } z > \delta$ and $\text{Im } z < -\delta$ we choose those levels in which $\Phi \to \mp 2i\delta$ if $\text{Im } z \to \pm\infty$, respectively. This results in cuts running along the lines $\text{Im} z = \pm\delta$, $\text{Re} z \leq 0$ and $\text{Re} z \geq \pi$, and corresponds to choosing $f_1 = f_2 = 0$ in [6]. Taking the logarithm of (2.4) it becomes:

$$N\Phi(v_\alpha, \gamma/2) = 2\pi I_\alpha + \sum_\beta \Phi(v_\alpha - v_\beta, \gamma), \tag{2.8}$$

where the I_α are half-odd integers and turn out to be very useful quantum numbers for characterizing the states with macroscopic number of spin waves [4]. The energy and the momentum in terms of the rapidities are

$$E = \frac{V}{a} \frac{\sinh \gamma}{2} \sum_\alpha \Phi'(v_\alpha, \gamma/2), \qquad Q = \sum_\alpha \Phi(v_\alpha, \gamma/2). \tag{2.9}$$

For technical reasons it is useful to restrict the real part of the rapidities to an interval of length π. We choose $0 < \text{Re}\, v_\alpha \leq \pi$, but note, that for our results it is important only, that the interval contains the point $\pi/2$ in its interior. Note also, that the wave function, the energy and the momentum are independent of the particular choice of this interval and branch cuts of $\Phi(z, \delta)$ (the latter modulo 2π).

B. The ground states

In the ground state(s) of the XXZ model with N even, there are $N/2$ rapidities ($S^z = 0$), all of them are real and all the I_j quantum numbers are *consecutive* half-odd integers satisfying $I_{j+1} - I_j = -1$ for $\lambda_{j+1} > \lambda_j$ [4]. (Here and in the following we denote the real rapidities by λ_j.) For large N the distribution of λ's can be well approximated by a smooth density $\sigma_0(\lambda)$ satisfying the linear integral equation [3–6]:

$$-\Phi'(\lambda, \gamma/2) = \sigma_0(\lambda) - \frac{1}{2\pi} \int\limits_{\Lambda}^{\Lambda+\pi} \Phi'(\lambda - \lambda', \gamma)\sigma_0(\lambda')d\lambda' , \tag{2.10}$$

$$\text{with} \quad N\Phi(\Lambda, \gamma/2) = 2\pi\left(I_1 + 1/2\right) + \sum_{j}^{N/2} \Phi(\Lambda - \lambda_j, \gamma) . \tag{2.11}$$

Eq.(2.10) can be solved by Fourier transformation leading to

$$\sigma_0(\lambda) = \frac{K}{\pi^2} \, \text{dn}\left(\frac{2K}{\pi}\lambda, k\right) \quad \text{with} \quad \frac{K'}{K} = \frac{\gamma}{\pi} , \tag{2.12}$$

where $\text{dn}(w)$ is the Jacobian elliptic function and K is the complete elliptic integral of the first kind with modulus k. (See Appendix A for notations and some properties of the elliptic functions.)

By means of this density the sum over j in (2.11) can be calculated, and the rapidity set can be reconstructed. One finds, that there are *two nonequivalent* sets of $N/2$ real λ_j's satisfying (2.8) with $I_{j+1} - I_j = -1$ [6]. These are:

$$\lambda_j = \frac{\pi}{2K}F\left(\frac{2\pi}{N}(j - I_0), k\right) , \quad I_j = 2I_0 - 1/2 - j , \quad j = 1, 2, \ldots N/2 \tag{2.13a}$$

for $N/2 =$ even and

$$\lambda_j = \frac{\pi}{2K}F\left(\frac{2\pi}{N}(j - 1/2 + I_0)), k\right) , \quad I_j = 1/2 - 2I_0 - j , \quad j = 1, 2, \ldots N/2 \tag{2.13b}$$

for $N/2 =$ odd, with $F(w, k)$ being the elliptic integral of first kind. The two sets are distinguished by I_0 taking the values of $1/2$ and 0, respectively. These two states of lowest energy are almost degenerate. The true ground state corresponds to $I_0 = 1/2$ and the difference between their energy is exponentially small in the length of the chain [8,7]. The leading term is:

$$\Delta E_0 = E(I_0{=}0) - E(I_0{=}1/2) = \frac{V}{a}\frac{\sqrt{8k'}}{\pi^{3/2}}\text{sh}\gamma K\frac{k_1^{N/2}}{N^{1/2}} , \tag{2.14}$$

where $k' = \sqrt{1 - k^2}$ and $\sqrt{k_1} = (1 - k')/k$. As ΔE_0 is exponentially small in the length of the chain and – as we shall see – all the other states are separated from the lowest two by a finite gap, we call these the two ground states or the physical vacua. The two vacua can be distinguished for instance by their momentum, Q and the eigenvalue, Σ of the operator (2.1) [7]:

$$Q(I_0, N) = 2\pi(I_0 - 1/2 + N/4)\,(\text{mod}\,2\pi) \tag{2.15}$$

$$\Sigma(I_0, N) = (-1)^{(2I_0-1+N/2)} . \tag{2.16}$$

C. The excited states

The states other than the two vacua are called excited states and the ones of not too high energy can be described in terms of some dressed particles. These are created by leaving holes in the sequence of the λ_j's, and by introducing complex rapidities. The smooth density $\sigma(\lambda)$ of the real rapidities is still a good approximation but (2.10) receives contributions from both the holes and the complex rapidities. When one solves for $\sigma(\lambda)$ we are left with a few equations relating the parameters of the excitations only [5,6]:

$$1 = \exp i\left\{ N\left(\text{am}\left(\frac{2K}{\pi}\theta_h, k\right) - \frac{\pi}{2}\right) - 2\pi I_0 + \sum_{b=1}^{n(\theta)}\left(\mathcal{F}(\theta_h - \theta_b, \gamma) - \frac{\pi}{2}\right) + \sum_{\alpha}^{n(\chi)}\Phi\left(\theta_h - \chi_\alpha, \frac{\gamma}{2}\right)\right\} \quad (2.17)$$

$$1 = \exp i\left\{ \sum_{h=1}^{n(\theta)}\Phi(\chi_\alpha - \theta_h, \gamma/2) - \sum_{\beta=1}^{n(\chi)}\Phi(\chi_\alpha - \chi_\beta, \gamma) - \pi\right\} \quad (2.18)$$

where

$$\mathcal{F}(x, \gamma) \equiv x + \sum_{m=1}^{\infty}\frac{e^{-\gamma m}}{m\cosh(\gamma m)}\sin(2mx), \quad (2.19)$$

and I_0 is again either $1/2$ or 0. The positions of the holes in the real-rapidity distribution are denoted by θ_h and their number is denoted by $n(\theta)$, whose parity is the same as that of N. The variables χ represent the set of complex rapidities z in the following way. If for a χ $|\text{Im}\chi| < \gamma/2$, it represents a close pair

$$z^\pm = \chi \pm i\gamma/2 + \mathcal{O}(e^{-N\Omega(z^+, \gamma)}) \quad (2.20a)$$

with $\Omega(z, \gamma)$ being a positive valued function of the order of unity (as long as $\gamma > 0$). If $|\text{Im}\chi| > \gamma/2$, the χ represents a wide root:

$$z = \chi + i\,\text{sgn}(\text{Im}\chi)\gamma/2. \quad (2.20b)$$

As the χs are either real, or form complex conjugated pairs, the typical complex rapidity configurations are the 2-strings (the two z represented by a real χ), the quartets (the four complex rapidities represented by a complex conjugated pair with $|\text{Im}\chi| < \gamma/2$), and wide pairs (corresponding to a complex conjugated pair of χs with $|\text{Im}\chi| > \gamma/2$). The total number of the complex parameters χ_α is denoted by $n(\chi)$.

The energy and the momentum of the excited states can be expressed in terms of the parameters of the holes only and are given as:

$$E = E_0 + \sum_h \epsilon(\theta_h) \quad \text{with} \quad \epsilon(\theta) = \frac{V}{a}\frac{K}{\pi}\sinh\gamma\,\text{dn}\left(\frac{2K}{\pi}\theta, k\right), \quad (2.21)$$

$$Q = Q(I_0, N) + \sum_h q(\theta_h) \quad \text{with} \quad q(\theta) = \text{am}\left(\frac{2K}{\pi}\theta, k\right) - \frac{1}{2}\pi, \quad (2.22)$$

with E_0 being the ground state energy, and $Q(I_0, N)$ is given in (2.15). The appearance of $Q(I_0, N)$ in the momentum suggests, that the $I_0 = 1/2$ resp. $I_0 = 0$ excited states should be considered as excitations above the $I_0 = 1/2$ resp. $I_0 = 0$ vacua. The relation between the functions $\epsilon(\theta)$ and $q(\theta)$ give the dispersion relation of the particles:

$$\epsilon(q) = \frac{V}{a}\frac{K}{\pi}\sinh\gamma\sqrt{1 - k^2\cos^2 q}\,. \tag{2.23}$$

The spin of the state characterized by a solution is

$$S^z = \frac{1}{2}n(\theta) - n(\chi)\,. \tag{2.24}$$

The states with $S^z < 0$ are obtained by flipping all spins, for example using the operator $\hat{\Sigma}$ of (2.1). The $S^z = 0$ BA eigenstates are expected to be eigenstates of this operation: otherwise certain points of the spectrum were twofold degenerated (with the two states connected by $\hat{\Sigma}$). This can happen *accidentally* but is not forced by the symmetry, as $\hat{\Sigma}$ has one-dimensional representations only. Recently Doikou and Nepomechi has proposed a formula for the parity of the $S^z = 0$ in the planar ($\rho < 1$) region [16]. Now we *conjecture*, that in the easy axis region we study, the spin reversal symmetry of the $S^z = 0$ states in analogy with their formula is given as

$$\Sigma = (-1)^\mu, \qquad \text{with} \qquad \mu = \frac{N}{2} + \frac{2}{\pi}\sum_{\alpha=1}^{N/2} v_\alpha\,. \tag{2.25}$$

We have not proved this formula, but performed numerical calculations to see this symmetry (Appendix E): solving the BA equations numerically for a number of $S^z = 0$ states has shown, that the wave function is either symmetric or antisymmetric under $\hat{\Sigma}$, and the value of Σ is correctly given by (2.25). In the large N limit (2.25) can be transformed into a simpler form using the density of the real roots [6]:

$$\mu = (2I_0 + 1) + \frac{N}{2} + \frac{2}{\pi}\left\{\sum_\alpha^{n(\chi)} \chi_\alpha - \frac{1}{2}\sum_h^{n(\theta)} \theta_h\right\} \pmod 2. \tag{2.26}$$

In summary, the structure of the states can be described as follows: the system has two vacua found in the N even system, one of them is symmetric, the other is anti-symmetric under (2.1), one of them has no (lattice) momentum, in the other this quantity is π. Above each vacuum there is a set of excited states which can be considered as scattering states of dressed particles. The momenta of the particles are quantized through a set of equations of the BA type (2.17-2.18) (HLBAE). The excitation energy is the sum of the energy contributions of the individual particles and the momentum is the sum of the contribution of the individual particles added to the momentum of the corresponding vacuum. The $S^z = 0$ states are eigenstates of the operation flipping all spins, and the eigenvalue of this operation is related to the chain length, quantum number I_0 and the rapidities describing the excitations.

III. THE SCALING LIMIT

A. The relativistic spectrum

Before constructing the scaling limit we note, that certain quantities (the momentum Q and the reflection Σ) depend on the parity of N (and $N/2$) but not on its magnitude. To make the procedure uniquely defined we carry out the $N \to \infty$ limit through Ns being integer multiples of four, in which case both in Q and Σ the N can be replaced by zero, and at the end of the section we discuss the consequences of other possible choices.

First, we show that (1.3) defines the limit in which the dispersion relation (2.23) describes relativistic particles. Note that the continuum momentum (P, p) is obtained from the lattice momentum (Q, q) after division by the lattice spacing a:

$$\epsilon^2 = \left(\frac{V}{a}\frac{K}{\pi}\sinh\gamma\right)^2 \left(1 - k^2\cos^2(ap)\right) \approx \left(\frac{VKk\sinh\gamma}{\pi}\right)^2 \left(\left(\frac{k'}{ka}\right)^2 + p^2\right). \tag{3.1}$$

Here we expanded $\cos(ap)$ about 1 and now observe that in the continuum limit ($a \to 0$) a relativistic dispersion relation emerges, provided $\gamma \to 0$ together with

$$\left(\frac{VKk\sinh\gamma}{\pi}\right) = 1 \quad \text{and} \quad \left(\frac{k'}{ka}\right) = M. \tag{3.2}$$

These requirements imply $V = 2/\pi$ and (1.3) (see Appendix A).

It will prove to be useful to introduce the scaled rapidity variable, ϑ which also makes the limiting procedure more transparent. It is defined in terms of the θ and γ as (see also (2.12)):

$$K + \vartheta \equiv K\frac{2\theta}{\pi}, \tag{3.3}$$

while the energy $\varepsilon(\vartheta) = \epsilon(\theta(\vartheta))$ and momentum $p(\vartheta) = \frac{1}{a}q(\theta(\vartheta))$ can be expressed as

$$\varepsilon(\vartheta) = \frac{2K\sinh\gamma}{\pi^2}\frac{k'}{a}\frac{1}{\mathrm{dn}(\vartheta)} \quad ; \qquad \frac{\sin(ap(\vartheta))}{a} = \frac{k'}{a}\frac{\mathrm{sn}(\vartheta)}{\mathrm{dn}(\vartheta)}. \tag{3.4}$$

In the scaling limit ϑ/K approaches 0 as $k \to 1$ which yields

$$\pi\frac{\theta - \pi/2}{\gamma} \to \vartheta \tag{3.5}$$

$$\varepsilon(\vartheta) \to \frac{k'}{a}\cosh(\vartheta) \quad ; \qquad p(\vartheta) \to \frac{k'}{a}\sinh(\vartheta), \tag{3.6}$$

in accord with $k'/a \to M = \text{constant}$.

The total momentum also contains a macroscopic quantity (2.22) and can be written in the continuum limit as,

$$P = \frac{Q(I_0)}{a} + \sum_h M \sinh(\vartheta_h) \,. \tag{3.7}$$

In view of (3.7) the two sectors corresponding to $I_0 = 1/2$ and $I_0 = 0$ become infinitely separated in momentum space lending a strong support to interpreting the $I_0 = 1/2$ resp. $I_0 = 0$ excited states as excitations above the corresponding ($I_0 = 1/2$ resp. $I_0 = 0$) vacua. Technically while (3.7) gives the true momentum for the $I_0 = 1/2$ sector in the $a \to 0$ limit, the continuum momentum of the $I_0 = 0$ sector needs to be redefined as

$$P = \frac{Q(I_0) + \pi}{a} + \sum_h M \sinh(\vartheta_h) \,. \tag{3.8}$$

As an alternative, we might keep both sets by redefining the lattice so that two sites form one elementary cell. In this case the lattice momenta 0 and π are equivalent, and the $Q(I_0)$ term can be dropped. In any case the energy and momentum measured relative to those of the corresponding vacuum are

$$E - E_0 = \sum_h M \cosh(\vartheta_h) \,, \qquad P = \sum_h M \sinh(\vartheta_h) \,. \tag{3.9}$$

B. The vacua in the scaling limit

The two lowest lying states are characterized purely by the density of rapidities, $\sigma_0(\lambda)$ whose behavior in the scaling limit is crucial for our arguments. The reason is that our considerations about the excited states are based on the existence of a non-vanishing background of rapidities (see also Appendix C). One can show that although $\sigma_0(\lambda)$ vanishes around $\pi/2$ (and diverges at 0 and π), the density of the ϑ's is non-vanishing as required. The density $\varrho_0(\vartheta)$ of the rescaled rapidities (3.3) can be expressed as

$$N\sigma_0\left(\frac{\pi}{2K}(\vartheta + K)\right) d\lambda = L\varrho_0(\vartheta)d\vartheta \,, \tag{3.10}$$

provided $d\vartheta = 2K d\lambda/\pi$ (here the l.h.s. is the number of λ's in the interval $\{\lambda, \lambda + d\lambda\}$ at $\lambda = \pi/2 + \pi\vartheta/2K$, and the r.h.s. gives the number of ϑ's in the interval $\{\vartheta, \vartheta + d\vartheta\}$), which leads to

$$\varrho_0(\vartheta) = \frac{1}{2\pi} M \cosh(\vartheta) \,. \tag{3.11}$$

The (1.5) difference in the energies of the two vacua is obtained directly by evaluating the scaling limit of (2.14). We have to note, however, that (2.14) is the leading term in a large N expansion, and as the expansion coefficients are functions of γ one has to see, if these neglected terms behave properly. In Appendix B we show that (1.5) is the leading term of a large L expansion of the same quantity and the neglected terms decay faster.

Finally we note, that the reflection symmetry is not effected by the limiting process, i.e. $\Sigma = \pm 1$ in the two states.

C. Secular equations in the scaling limit

We define the secular equations of the limiting theory as the SL of the HLBAE of the lattice model. This limit exists but if it is meaningful is a delicate question: deriving the HLBAE two approximations has been used, integrating instead of summing over the rapidities, and neglecting the exponentially small corrections to the close pairs. These have to be justified even in the SL. Replacing the sums with integrals require that the density of rapidities does not vanish in the scaling limit (see the previous section) more over it can be shown that the error introduced remains negligible. Also the corrections to the close pairs remain exponentially small even in the SL as it is shown in Appendix C.

To derive the scaling limit of the HLBA we first observe, that the energy and momentum diverge unless all θ scale to $\pi/2$ as $\gamma \to 0$ (i.e. all ϑ are finite). In this case we expect also some of the χ's to scale to $\pi/2$. For convenience we introduce their rescaled versions as

$$\kappa = \pi \frac{\chi - \pi/2}{\gamma} \tag{3.12}$$

It is straightforward to determine how the functions am , $\mathcal{F}$ and Φ behave in the scaling limit:

$$\frac{1}{a}\left(\text{am}(\vartheta + K) - \frac{\pi}{2}\right) \xrightarrow{\gamma \to 0} p(\vartheta) = M \sinh \vartheta, \tag{3.13}$$

$$\mathcal{F}\left(\frac{\Delta\vartheta\gamma}{\pi}, \gamma\right) \xrightarrow{\gamma \to 0} \phi\left(\frac{\Delta\vartheta}{\pi}\right), \tag{3.14}$$

$$\Phi\left(\frac{\Delta\kappa\gamma}{\pi}, \frac{\gamma}{m}\right) \xrightarrow{\gamma \to 0} \pi - 2\tan^{-1}\left(\frac{\Delta\kappa}{\pi/m}\right), \tag{3.15}$$

$$\Phi\left(\Delta\chi, \frac{\gamma}{m}\right) \xrightarrow{\gamma \to 0} 0 \qquad \text{if } \Delta\chi \to \text{finite.} \tag{3.16}$$

After substituting the above expressions to the HLBA equations, their scaling limit is obtained. The complex rapidities which remain at finite distance from $\pi/2$ drop out from the equations, although they do affect S^z of the state as we will see in the next section. The equations for the holes and the complex parameters approaching $\pi/2$ become

$$Lp(\vartheta_h) = 2\pi(\mathcal{I}_h + I_0) - \sum_{l}^{n(\vartheta)} \phi\left(\frac{\vartheta_h - \vartheta_l}{\pi}\right) + \sum_{\alpha}^{n(\kappa)} 2\tan^{-1}\frac{\vartheta_h - \kappa_\alpha}{\pi/2}, \tag{3.17}$$

$$\mathcal{I}_h = \frac{1}{2}\left(\frac{n(\vartheta) - 2n(\kappa)}{2}\right) (\mathrm{od}\,1), \tag{3.18}$$

$$\sum_{h}^{n(\vartheta)} 2\tan^{-1}\frac{\kappa_\alpha - \vartheta_h}{\pi/2} = 2\pi\mathcal{J}_\alpha + \sum_{\beta}^{n(\kappa)} 2\tan^{-1}\frac{\kappa_\alpha - \kappa_\beta}{\pi}, \tag{3.19}$$

$$\mathcal{J}_\alpha = \left(\frac{n(\kappa) - n(\vartheta) + 1}{2}\right) (\mathrm{od}\,1). \tag{3.20}$$

It is to be noted, that the quantum number I_0 directly appears in the secular equations of the excited states. As we have discussed before, we interpret the two sets of solutions distinguished by the value of I_0 as the excited states of the corresponding vacua.

D. Multiplet structure of the states

A few simple solution of the above secular equations are given and the structure of solutions is discussed in Appendix D. Here we discuss the degenerations in the spectrum developing in the SL. As both the energy and the momentum depend on the rapidities of the holes only, we expect degenerate multiplets corresponding to solutions differing in their χ content only. As (3.17) and (3.19) contain both the ϑ and κ sets the solutions becoming degenerate can differ in the χ's not scaling to $\pi/2$ (and which disappeared from the equations).

In Appendix D we discuss the types of different solutions of the HLBAE (2.18) and relate them to the solutions of (3.19). We argue, that at a given number of holes, $n(\vartheta)$, each having the form $\theta_h = \gamma\vartheta_h/\pi + \pi/2$, to any number $0 \le n(\kappa) \le n(\vartheta)/2$ (2.18) have $n(\vartheta)/2 - n(\kappa) + 1$ different solutions in which $n(\kappa)$ of the χs scale to $\pi/2$ like $\chi_\alpha = \gamma\kappa_\alpha/\pi + \pi/2$ with the common κ_α set being one solution of (3.19). These solutions differ in the number of χs not scaling to $\pi/2$. It is clear than, that the number of different *solutions* of (2.17-2.18) becoming degenerated in the SL and given by one solution of the system (3.17-3.19) is $n(\vartheta)/2 - n(\kappa) + 1$. The number of χs not scaling to $\pi/2$ can be $0, 1, \ldots, n(\vartheta)/2 - n(\kappa)$, and the corresponding states have spins $S^z = n(\vartheta)/2 - n(\kappa), \ldots, 1, 0$ respectively. Completing this set of states with the states $S^z = -1, -2, \ldots, -(n(\vartheta)/2 - n(\kappa))$ which are obtained by flipping all spins we see, that the degeneration of the *states* given by one solution of the equations (3.17-3.19) is $n(\vartheta) - 2n(\kappa) + 1$ This corresponds exactly to SU(2) multiplets: the states are characterized by two quantum number $l = n(\vartheta)/2 - n(\kappa)$ and $l \ge m \ge -l$, and the m does not influence the energy. This way we may consider the particles as 'spin' 1/2 particles forming SU(2) eigenstates with quantumnubers l and m. (This picture will be refined when discussing the spin reversal symmetry.) Apparently the equations (3.17) and (3.19) give the $l = m$ states directly (as usual in SU(2) BA systems), and to have the complete description of the $m < l$ states one should find the χs not scaling to $\pi/2$, nevertheless these parameters do not enter

into the spectrum. This interpretation is consistent with the expectation for the number of different solutions: the number of nonequivalent solutions of (3.19) given by (D22) is just the number of different $l = n(\vartheta)/2 - n(\kappa)$, $l = m$ states of $n(\vartheta)$ spin $1/2$ particles. One has to note, however, that while the quantumnumber m coincides with the S^z of the system, it can not be seen, if $S^2 = l(l+1)$ or the quantity given by $l(l+1)$ is different from the square of the total spin. Note also, that this $SU(2)$ symmetry develops in the SL, but it is not present in the initial model.

E. The two particle scattering matrix

The two particle scattering phase shifts of the physical particles can be reconstructed from the secular equations. The method for this is based on the idea, that the deviations of the particle momenta from the free values can be interpreted as the phase shifts of the particles scattering on each other [17–19]. Consider a two particle scattering-state on a ring. If the momenta are p_1 and p_2 and the particles obey twisted boundary conditions with a twisting angle φ ($\varphi = 0$ for periodic and it is π for the antiperiodic boundary condition), then the boundary condition requires $Lp_1 + \delta_{12} = 2\pi n_1 + \varphi$ and $Lp_2 - \delta_{12} = 2\pi n_2 + \varphi$ with n_1 and n_2 integers and δ_{12} being the phase shift [20,21]. Writing the equations (3.17) in this form the phase shifts can be found. A weakness of this procedure is that in most of the interesting cases φ is not known (as the boundary conditions for the dressed particles can differ from those of the bare ones). The twisting could be seen directly in the equation describing one single particle. Since, however, the parity of the particle number is usually determined by that of the number of elements in the chain, the sectors with even and odd number of particles are disjoint in the sense, that a given system represents either one or the other, and one can not see, if the twisting is the same in both sectors. (A trivial example for particle number dependent boundary conditions is given by a spin chain where the spin waves are described by Fermions through a Jordan-Wigner transformation.) Now we *suppose* periodic boundary conditions, i.e. $\varphi = 0$, but we keep in mind, that our results hold up to a rapidity independent constant (see also [22]).

A triplet state of two particles is described by two ϑs and no κs. For such a state Eq. (3.17) yields

$$\delta_{12}^{tr} = \phi\left(\frac{\Delta\vartheta}{\pi}\right) + \pi - 2\pi I_0\,, \quad \Delta\vartheta = \vartheta_1 - \vartheta_2\,, \tag{3.21}$$

where the π comes from the parity prescription for the parameters $\mathcal{I}_h$. A singlet state is characterized in addition to ϑ_1 and ϑ_2 by a κ for which Eq. (3.19) yields $\kappa = (\vartheta_1 + \vartheta_2)/2$. For this case the above reasoning leads to

$$\delta_{12}^{s} = \phi\left(\frac{\Delta\vartheta}{\pi}\right) - 2\tan^{-1}\left(\frac{\Delta\vartheta}{\pi}\right) - 2\pi I_0\,. \tag{3.22}$$

From the phase shifts (3.21) and (3.22) the two particle S-matrix can be given as

$$\hat{S}(\Delta\vartheta) = -\exp\left\{i\left(2\pi I_0 + \phi\left(\frac{\Delta\vartheta}{\pi}\right)\right)\right\}\left(\hat{P}_{tr} + \frac{\Delta\vartheta + i\pi}{\Delta\vartheta - i\pi}\hat{P}_s\right),\qquad(3.23)$$

where $\hat{P}_{tr}$ and $\hat{P}_s$ are the projectors on the triplet resp. singlet subspace of the two spins. This S-matrix is consistent with the SU(2) symmetry of the limiting model detected in the degenerations.

The scattering matrices (3.23) of the two sets of solutions differ in an overall sign due to the appearance of I_0. To decide, if this difference is significant in the sense that it has physical consequences regarding the nature of the particles in the two sets of states, needs further considerations.

F. Spin reversal symmetry

Finally we have to discuss the question of the symmetry of the excited states with respect to $\hat{\Sigma}$ of (2.1). The two vacua are eigenstates with eigenvalues ± 1. As we have mentioned it already, checking numerically a number of $S^z = 0$ excited states of the finite lattice model (with finite γ) we have seen, that these states are also eigenstates of $\hat{\Sigma}$ with the eigenvalue Σ given by (2.25-2.26), but on that level we could not see an easy rule for the eigenvalues. We have found some indications, however, that there is a correlation between the spin-structure of the dressed particles and the spin reversal symmetry. We have examined a number of solutions of different types with two holes and one close pair of the $N = 20$ lattice. It has been found that in accordance with Appendix D the solutions form two classes: in class 1 the real part of the close pair is in the vicinity of $(\theta_1 + \theta_2)/2$ while in class 2 it is in the neighborhood of $(\theta_1 + \theta_2 \pm \pi)/2$. In case of $I_0 = 1/2$ the solutions in class 1 are symmetric and the others, in class 2, are antisymmetric under $\hat{\Sigma}$ while just the opposite is true for $I_0 = 0$ (Appendix E).

In the scaling limit the solutions of class 1 resp. class 2 satisfy (D5) resp. (D6) representing singlet and triplet states. We believe that while increasing the value of N the symmetry properties do not change, and we conclude, that in this type of states the singlets have the symmetry of the vacuum, and the symmetry of the triplets is the opposite. Accepting the validity of (2.25) (and so that of (2.26)) we can have Σ also for the $n(\theta) = 4$, $n(\chi) = 2$ states. Taking the results of Appendix D we find that

$$\Sigma = \Sigma(I_0, N, l) = (-1)^{(2I_0 - 1) + N/2 - l} = \Sigma(I_0, N)(-1)^l \qquad (3.24)$$

with $l = 0$, 1, 2, corresponding to singlet, triplet and quintuplet configurations, respectively. We think, this is an indication, that (3.24) is generally true.

If we perceive the reflection symmetry as the product of the symmetry of the vacuum and that of the spin structure of the excitations, the latter should be

$$\Sigma = (-1)^l \tag{3.25}$$

i.e. we have to consider now the singlets to be symmetric and the triplets anti-symmetric. This is just the opposite as in the case of two ordinary SU(2) spins, and resembles a q-deformed SU(2) structure at $q = -1$ [23]. It can be shown in general [24], that defining the spin operators for a system of particles as

$$\sigma_2^{\pm} = \sum (-1)^{(j-1)} \sigma_j^{\pm} \qquad \sigma_2^z = \sum \sigma_j^z \tag{3.26}$$

(with σ_j being the Pauli matrices acting on the spin labeled by j) results an SU(2) structure (equivalent to a q-deformed SU(2) at $q = -1$) in which the $S^z = 0$ members of the multiplets have the eigenvalue (3.25). Particularly in the case of two spins the $\sigma_2^z = 0$ member of the triplet is $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$, while the singlet is $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$.

We have to note, that the symmetry properties of the limiting theory are in strong analogy with that of an ordinary XXX Heisenberg chain. The eigenstates of this model are S^2 and S^z eigenstates with eigenvalues $l(l+1)$ and m, respectively, with $l \geq |m|$ integers if N is even. It is known, that the $m = 0$ states of an even number of spins are eigenstates of $\hat{\Sigma}$ with an eigenvalue analogous to (3.24)

$$\Sigma_{XXX} = (-1)^{N/2-l} . \tag{3.27}$$

This is a consequence of the fact, that any such state can be given as linear combination of states built up as products of l triplet and $N/2 - l$ singlet pairs [26]. The ground state (the vacuum) is singlet, thus all the singlet excitations have the same symmetry as the vacuum and the triplets have the opposite. Thus we may consider the degenerated states as forming normal SU(2) multiplets of the N spin of a the chain, but equally well we may think of these states as the above described multiplets of the *dressed particles*, and all this is a direct consequence of the SU(2) symmetry of the XXX Hamiltonian. In the case of our limiting theory there is no Hamiltonian, and the symmetry is recognized in an indirect way: the SL is taken through states not showing the SU(2) structure, but this symmetry appears, as degenerations characteristic for this symmetry develop. Now we see, that the reflection symmetry properties complete this structure.

G. Boundary conditions in the limiting theory

In the previous part of the Section we supposed, that the $N \to \infty$ limit is taken through integer multiples of four ($N = 4\mathcal{N}$). Actually in order to be able to define the limit of the

momentum uniquely and to obtain the parameters in the limiting equations together with the properties of the excited states properly defined, we have to fix the way of the $N \to \infty$ limit, but we have four different possibilities: $N = 4\mathcal{N} + \nu$, $\mathcal{N} \to \infty$ and $\nu = 0, 1, 2, 3$. In these cases (2.15) and (3.24) are

$$Q(I_0, N) \equiv Q(I_0, \nu) = 2\pi(I_0 - 1/2 + \nu/4) \,, \tag{3.28}$$

$$\Sigma(I_0, N, l) = \Sigma(I_0, \nu, l) = (-1)^{(2I_0-1)+\nu/2+l} \tag{3.29}$$

(the latter applying for $\nu = 0, 2$ only, as $\hat{\Sigma}$ has no eigenstates among the BA states of the N=odd chain). Now we want to investigate the $\nu = 1, 2, 3$ cases.

First consider the $\nu = 2$ case. Now we have to define the momentum in the continuum limit as

$$P = \frac{Q(I_0, 2) - \pi}{a} + \sum_h M \sinh(\vartheta_h) \tag{3.30}$$

or

$$P = \frac{Q(I_0, 2)}{a} + \sum_h M \sinh(\vartheta_h) \tag{3.31}$$

to pick the $I_0 = 1/2$ or $I_0 = 0$ states respectively. In this procedure all the results are exactly the same as before, except those concerning the reflection symmetry (and as we shall see the S-matrices must be affected too). Now, according to (3.29) ((2.16)) the secular equations (3.17) with $I_0 = 0$ describe the excitations of the reflection symmetric vacuum, and $I_0 = 1/2$ gives the excitations of the antisymmetric one (just the opposite as for $\nu = 0$). The nature of the excitations should be given by the symmetry of the corresponding vacuum, and the secular equations are the quantizations of the momenta of the individual particles. Now the fact, that the same set of particles (say a singlet pair above one of the vacua) may obey two different quantization condition, can be interpreted so that in the two cases the *boundary conditions* are different: if they are periodic in one case, they are anti-periodic in the other (the difference in the twisting is π). The two particle scattering matrices given by (3.23) are obtained supposing periodic boundary conditions. From the very same secular equations but supposing anti-periodic boundary conditions the same S-matrices multiplied by (-1) are obtained, i.e. interpreting the limiting theories for $\nu = 0$, $I_0 = 0(1/2)$ and $\nu = 2$, $I_0 = 1/2(0)$ as being the same ones but with different boundary conditions, is consistent with the S-matrices obtained. It is also consistent with the reflection symmetry of the $S^z = 0$ states: we checked numerically (for $N = 18$), that also in the $\nu = 2$ case the two particle states scaling to singlets have the same symmetry as the vacuum.

Now it is not hard to see, that with slight modifications of the prescription of the limiting process for the momentum we can consistently define the scaling limit also for $\nu = 1$ and $\nu = 3$. This way, together with the two possible values of I_0 we can obtain four cases.

Also these are described by the equations (3.17-3.19) but with the restriction, that $n(\vartheta)$ must be odd. Also in these cases the eigenstates form SU(2) multiplets corresponding to the different combinations of $n(\vartheta)$ spins of length 1/2 and Eqs. (3.17-3.19), just as in the previously discussed cases, give directly the $l = m = n(\vartheta)/2 - n(\kappa)$ states. We interpret these four sets of states as the excited states with odd numbers of particles above the two vacua, both with two different boundary conditions. Unfortunately, as none of these sets contain the corresponding vacuum, we can not tell, which values of the parameters I_0 and ν correspond to which vacuum. Now, however, the boundary conditions can be seen directly: the equations of the one particle states read

$$Lp(\vartheta) = 2\pi(I_0 + 1/4 + \text{integer}) \tag{3.32}$$

indicating, that these particles obey twisted boundary conditions with twisting angle $2\pi(I_0 + 1/4)$ [20,21].

A possible explanation of the fact that for different νs we get different limits can be the following. In many cases in order to define the local operators in a naive continuum limit one has to group the lattice points into elementary cells of certain length [25,10,11], which will be the points of the continuum. If the chain length is not an integer multiple of the size of the elementary cell, necessarily boundary terms occur in the limit which can cause a rapidity independent shift in the phase (twisting) as a particle is taken around the ring. Now it seems, that the size of the elementary cell is four, and so there are four different possibilities as far as the 'surface terms' are concerned. In addition to this the number of particles must have the same parity as the chain-length (the vacua are spinles, while the particles are spin 1/2 particles). These together add up to give the variety of sectors discussed above.

IV. ACKNOWLEDGMENT

APPENDIX A:

In this appendix we list those definitions and properties of the elliptic integrals and functions [14] we use in the bulk of the paper.

$$F(\phi, k) \equiv \int_0^\phi \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}} \tag{A1}$$

$$K \equiv \int_0^{\frac{\pi}{2}} (1 - k^2 \sin^2 \alpha)^{-\frac{1}{2}} d\alpha \tag{A2}$$

$$K' \equiv \int_0^{\frac{\pi}{2}} (1 - k'^2 \sin^2 \alpha)^{-\frac{1}{2}} d\alpha \tag{A3}$$

$$k^2 + k'^2 = 1 \tag{A4}$$

$$k_1 = \frac{(1 - k')^2}{k^2} \tag{A5}$$

$$K' = \frac{\pi}{2}(1 + \frac{1}{4}k'^2 + \mathcal{O}(k'^4)) \tag{A6}$$

$$K = \log \frac{4}{k'} + \frac{1}{4}(\log \frac{4}{k'} - 1)k'^2 + \mathcal{O}(k'^{4-0}) \tag{A7}$$

$$\frac{K'}{K} = \frac{\pi}{2 \log \frac{4}{k'}}(1 + \frac{k'^2}{4 \log \frac{4}{k'}} + \mathcal{O}(k'^{4-0})) \tag{A8}$$

$$\gamma = \pi \frac{K'}{K} \tag{A9}$$

$$k' = 4e^{-\frac{\pi^2}{2\gamma}} - 16e^{\frac{3\pi^2}{2\gamma}} + \dots \tag{A10}$$

$$K = \frac{\pi^2}{2\gamma} + \frac{2\pi^2}{\gamma}e^{-\frac{\pi^2}{\gamma}} + \dots \tag{A11}$$

$$K' = \frac{\pi}{2}(1 + 4e^{-\frac{\pi^2}{\gamma}} + \dots) \tag{A12}$$

$$\frac{\partial \operatorname{am}(u, k)}{\partial u} = \operatorname{dn}(u, k) \tag{A13}$$

$$\frac{\partial \operatorname{sn}(u, k)}{\partial u} = \operatorname{cn}(u, k)\operatorname{dn}(u, k) \tag{A14}$$

$$\frac{\partial \operatorname{cn}(u, k)}{\partial u} = -\operatorname{sn}(u, k)\operatorname{dn}(u, k) \tag{A15}$$

$$\frac{\partial \operatorname{dn}(u, k)}{\partial u} = -k^2\operatorname{sn}(u, k)\operatorname{cn}(u, k) \tag{A16}$$

$$\operatorname{sn}(u, k) = \sin \operatorname{am}(u, k) \tag{A17}$$

$$\operatorname{dn}^2(u, k) = 1 - k^2\operatorname{sn}^2(u, k) \tag{A18}$$

$$\operatorname{dn}(\pm K + u, k) = k'\frac{1}{\operatorname{dn}(u, k)} \tag{A19}$$

$$\operatorname{cn}(\pm K + u, k) = \mp k'\frac{\operatorname{sn}(u, k)}{\operatorname{dn}(u, k)} \tag{A20}$$

$$F(\phi, k) \xrightarrow{k \to 1} \log \tan \left(\frac{\phi}{2} + \frac{\pi}{4}\right) \tag{A21}$$

$$\operatorname{sn}(u) \xrightarrow{k \to 1} \tanh(u) \tag{A22}$$

$$\operatorname{dn}(u) \xrightarrow{k \to 1} \frac{1}{\cosh(u)} \tag{A23}$$

APPENDIX B:

Here we analyze the behavior of the vacuum rapidity sets in the SL and give a derivation of the energy difference between the two vacua.

First one can see, that if $\gamma \to 0$ at fixed N the number of rapidities is fixed and they condense at the ends of the interval $\{0, \pi\}$, (except the $\lambda = \pi/2$ element in one of the sets), and the interior of the interval becomes empty. In this limit if the rapidities are rescaled $x = \lambda/\gamma$ (if $\lambda < \pi/2$) and $x = (\lambda - \pi)/\gamma$ (if $\lambda > \pi/2$), the rapidity sets of the singlet ground-state resp. the lowest energy triplet state of the XXX chain are recovered. If however, the $\gamma \to 0$ limit is performed under the condition (1.4), although most of the rapidities condense at 0 and π but the interior of the interval will not become empty, and the properly scaled rapidities have a finite density at $\pi/2$, as discussed in III B. This is an important condition for the applicability of the method deviced by [7] to calculate the energy difference of the two vacua.

The expression for the energy difference of the two vacua given in [7] gives the leading terms of a double series of the powers of $1/\sqrt{N}$ and $k_1^{N/2}$ in which the coefficients are functions of γ. If $\gamma \to 0$ at fixed N this expansion breaks down as $k_1 \to 1$ and also the expansion coefficients may explode. (While the expansion breaks down, the true energy difference will be $\propto 1/N$.) For this reason one has to be very careful, and if our (1.3-1.4) limit is performed, the limit is to be calculated in all terms. Evaluating the terms presented in [7] indicates that the limiting energy difference is indeed a function of L (just as (1.5) is). In the following we present a simple derivation of (1.5) in which one can see, that the neglected terms decay faster.

According to [7] the energy difference is given by

$$\Delta E \simeq -4\mathrm{Re} \int_{0+i\gamma/2}^{\pi+i\gamma/2} \epsilon(\theta)\sigma_0(\theta)e^{iNq(\theta)}d\theta \tag{B1}$$

where $\epsilon(\theta)$, $\sigma_0(\theta)$ and $q(\theta)$ are given by (2.21), (2.12) and (2.22) respectively, and the neglected terms are of the type

$$\int_{0+i\gamma/2}^{\pi+i\gamma/2} f(\theta)e^{iNnq(\theta)}d\theta \qquad (n \geq 2 \text{ integer}). \tag{B2}$$

The function $q(\theta + i\gamma/2)$ has an extremum at $\theta = \pi/2$, and we can use the saddle-point method. Introducing the variable ϑ of (3.3) we find [15], that around $\theta = \pi/2$

$$q(\theta + i\gamma/2) = \frac{1}{i}\ln\left(\frac{1 - k'}{k}\right) + \frac{ik'}{2}\vartheta^2. \tag{B3}$$

i.e. (as $k'/a \to M$):

$$iNq(\theta + i\gamma/2) = N \ln\left(\frac{1-k'}{k}\right) - \frac{LM}{2}\vartheta^2 \,. \tag{B4}$$

It is clear, that for large L to pick up the leading contribution we may use this expression in the integral, and we may expand also the ϵ and σ in terms of ϑ. This leads to

$$\Delta E \simeq \frac{8NK\mathrm{sh}\gamma(k')^2}{a\pi^3}\left(\frac{1-k'}{k}\right)^N \int_0^\infty \vartheta^2 e^{-LM\vartheta^2/2}d\vartheta \tag{B5}$$

which yields (1.5) indeed. We can also see, that the neglected terms (B2) are

$$\propto e^{-nLM} \tag{B6}$$

i.e. for large enough size L the energy difference decays as (1.5).

APPENDIX C:

In the bulk of the paper we derived the secular equations of the limiting model as the limit of the secular equations of the lattice model. This procedure is meaningful only if the approximations used to derive the original equations remain valid in the SL too.

One of the approximations applied is the replacement of summations by integrals. The error introduced this way can be calculated by the method of [7], and as it leads to the evaluation of formulae of the type (B1), we may conclude, that this type of corrections are exponentially small in LM.

Another subtle point of the derivation of the higher level BA equations is the representation of each close pair by a single number, namely by its center. As it is known, the close pairs of the original rapidity set are defined by those χ solutions of the higher level BA equations, for which $\mathrm{Im}\chi < \gamma/2$ through the rule (2.20a). This approximation is allowable, if $\exp\{-N\Omega\}$ will remain small in the SL. Now we show, that this is indeed the case. Actually according to [5]

$$\ln\left|z^+ - z^- - i\gamma\right| \simeq -N\Omega(z^+, \gamma)\,, \tag{C1}$$

with

$$\Omega(z^+, \gamma) = \frac{1}{2}\sum_{m=-\infty}^{\infty} \ln\left\{\frac{\cosh\frac{\pi}{\gamma}(x + m\pi) + \sin(\frac{\pi}{\gamma}y)}{\cosh\frac{\pi}{\gamma}(x + m\pi) - \sin(\frac{\pi}{\gamma}y)}\right\}\,, \quad (z^+ = x + iy, \quad y \le \gamma)\,. \tag{C2}$$

In studying the behavior of the Ω one has to distinguish three cases (see Appendix D):

1. The x scales to 0 or π:

$$x = \begin{cases} \xi\gamma/\pi & \xi \geq 0 \\ \pi + \xi\gamma/\pi & \xi < 0 \end{cases} \tag{C3}$$

In this case, as also $y \propto \gamma$, $\Omega \to$ finite, and $N\Omega \to \infty$, i.e. $z^+ - z^- - i\gamma \to 0$ exponentially fast in N.

2. Another possibility is, that x is finite but $x \nrightarrow \pi/2$. It is not hard to see, that in this case

$$N\Omega \simeq \frac{LM}{2} e^{\frac{\pi}{\gamma}|\frac{\pi}{2} - x|} \sin(\frac{\pi}{\gamma}y) . \tag{C4}$$

Also this expression diverges as $\gamma \to 0$.

3. Finally, if x scales to $\pi/2$, i.e. $x = \pi/2 + \gamma \mathrm{Re}\kappa/\pi$ (and $y = \gamma \mathrm{Im}\kappa/\pi + \gamma/2$ with $\mathrm{Im}\kappa < \pi/2$), one finds, that

$$N\Omega = LM \cosh(\mathrm{Re}\kappa) \cos(\mathrm{Im}\kappa) . \tag{C5}$$

The close pairs described as case 1. and 2. form exact 2-strings or quartets in the scaling limit, but they disappear from the secular equations. Nevertheless they play an important role in determining the multiplet structure of the states. The close pairs of case 3. are in strong analogy with the close pairs of a finite lattice model in the sense, that they form exact 2-strings or quartets in the infinite size limit only. As, however, the deviation is exponentially small in the size of the system, the representation of the close pairs by their centers is justified.

APPENDIX D:

In this Appendix we discus the structure of the solutions of (2.18), what we write in the form

$$\prod_{h=1}^{n(\theta)} \frac{\sin(\chi_j - \theta_h + i\gamma/2)}{\sin(\chi_j - \theta_h - i\gamma/2)} = -\prod_{k=1}^{n(\chi)} \frac{\sin(\chi_j - \chi_k + i\gamma)}{\sin(\chi_j - \chi_k - i\gamma)} . \tag{D1}$$

We give an account of their different types, and relate their $\gamma \to 0$ limit to the solutions of (3.19)

$$\prod_{h=1}^{n(\vartheta)} \frac{\kappa_j - \vartheta_h + i\pi/2}{\kappa_j - \vartheta_h - i\pi/2} = -\prod_{k=1}^{n(\kappa)} \frac{\kappa_j - \kappa_k + i\pi}{\kappa_j - \kappa_k - i\pi} . \tag{D2}$$

In this analysis we shall keep the scaled rapidities ϑ_h fixed while seeking for solutions to χ_j as a perturbative expression in γ:

$$\theta_h = \frac{\pi}{2} + \frac{\gamma}{\pi}\vartheta_h\,, \qquad \chi_j = \chi_{j0} + \frac{\gamma}{\pi}x_j + \mathcal{O}(\gamma^2). \tag{D3}$$

Note that $\chi_{j0} = \frac{\pi}{2}$ and $x_j = \kappa_j$ for the rapidities which scale to $\pi/2$ as was defined in (3.12).

We label the types of solutions by three numbers $\{H, l, m\}$. In this label the first number is $H = n(\vartheta)$, the number of particles, the last one is $m = S^z = n(\vartheta)/2 - n(\chi)$, i.e. the spin of the corresponding state, and the middle one is given by $l = n(\vartheta)/2 - n(\kappa)$. This quantum number is specified in the bulk of the paper, when the multiplet structure is discussed. (Note, that $l - m = n(\chi) - n(\kappa) \geq 0$ is the number of χs *not* scaling to $\pi/2$.)

We analyze in detail the states with $n(\vartheta)$ being equal to two and four.

States with $n(\vartheta) = 2$

$n(\chi) = 0$

The simplest such state is the one, in which there is no χ. The label of this state is $\{2, 1, 1\}$.

$n(\chi) = 1$

The equation

$$\prod_{h=1,2} \frac{\sin(\chi - \theta_h + i\gamma/2)}{\sin(\chi - \theta_h - i\gamma/2)} = 1 \tag{D4}$$

has two solutions:

$$\chi = \frac{\theta_1 + \theta_2}{2}\,, \tag{D5}$$

$$\chi = \frac{\theta_1 + \theta_2}{2} - \frac{\pi}{2}\,(\mathrm{mod}\,\pi)\,. \tag{D6}$$

In the first χ scales to $\pi/2$:

$$\chi = \frac{\gamma\kappa}{\pi} + \frac{\pi}{2} \tag{D7}$$

with

$$\kappa = \frac{\vartheta_1 + \vartheta_2}{2}\,, \tag{D8}$$

in the second not. According to our notations the two solutions are labeled by $\{2,0,0\}$ and $\{2,1,0\}$, respectively.

Note, that in the SL the $\{2,1,1\}$ and $\{2,1,0\}$ solutions pairwise become degenerate, as the equations fixing the ϑ_h's will be exactly the same for them.

States with $n(\vartheta) = 4$

$n(\chi) = 0$

The simplest state is the one with no χ. This is labeled by $\{4,2,2\}$.

$n(\chi) = 1$

The states with one χ are characterized by the four nonequivalent solutions of the equation

$$\prod_{h=1}^{4} \frac{\sin(\chi - \theta_h + i\gamma/2)}{\sin(\chi - \theta_h - i\gamma/2)} = 1 \,. \tag{D9}$$

Three of these are well approximated (in the $\gamma \to 0$ limit are exactly given) by the three (real and finite) solutions of the equation

$$\prod_{h=1}^{4} \frac{\chi - \theta_h + i\gamma/2}{\chi - \theta_h - i\gamma/2} = 1 \,, \qquad \left(\prod_{h=1}^{4} \frac{\kappa - \vartheta_h + i\pi/2)}{\kappa - \vartheta_h - i\pi/2)} = 1 \,, \right) \tag{D10}$$

and one χ is given in leading order by

$$\chi = \frac{1}{4} \sum_{h=1}^{4} \theta_h - \frac{\pi}{2} \,(\mathrm{mod}\,\pi) = \frac{\gamma}{4\pi} \sum_{h=1}^{4} \vartheta_h \,(\mathrm{mod}\,\pi) \,. \tag{D11}$$

The first three solutions have the label $\{4,1,1\}$, while the fourth is $\{4,2,1\}$.

$n(\chi) = 2$

The solutions for the equations of two χs

$$\prod_{h=1}^{4} \frac{\sin(\chi_j - \theta_h + i\gamma/2)}{\sin(\chi_j - \theta_h - i\gamma/2)} = - \prod_{k=1}^{2} \frac{\sin(\chi_j - \chi_k + i\gamma)}{\sin(\chi_j - \chi_k - i\gamma)} \tag{D12}$$

are of three type. The set of first type coincides in the $\gamma \to 0$ limit with the set of solutions of the equations

$$\prod_{h=1}^{4} \frac{\chi_j - \theta_h + i\gamma/2}{\chi_j - \theta_h - i\gamma/2} = -\prod_{k=1}^{2} \frac{\chi_j - \chi_k + i\gamma}{\chi_j - \chi_k - i\gamma}, \tag{D13}$$

i.e. they are given through (D7) by the solutions of

$$\prod_{h=1}^{4} \frac{\kappa_j - \vartheta_h + i\pi/2}{\kappa_j - \vartheta_h - i\pi/2} = -\prod_{k=1}^{2} \frac{\kappa_j - \kappa_k + i\pi}{\kappa_j - \kappa_k - i\pi}. \tag{D14}$$

These solutions are labeled by $\{4, 0, 0\}$. (These solutions can be given even analytically. The two nonsingular solutions (i.e. in which the roots are not at the singularities of the equations) are of the form

$$\kappa_{1,2} = \frac{1}{4} \sum_h^4 \vartheta_h \pm \Delta\vartheta, \tag{D15}$$

with $(\Delta\vartheta)^2$ given by

$$\left((\Delta\vartheta)^2\right)_{1,2} = \frac{1}{6}\left(-\mathcal{B} \pm \sqrt{\mathcal{B}^2 + 12\mathcal{C}}\right) \tag{D16}$$

with

$$\mathcal{B} = 2\left(\frac{\pi}{2}\right)^2 + \sum_{h<h'}(\delta\vartheta_h)(\delta\vartheta_{h'}), \quad \mathcal{C} = \left(\frac{\pi}{2}\right)^4 - 2\left(\frac{\pi}{2}\right)^2 \sum_{h<h'}(\delta\vartheta_h)(\delta\vartheta_{h'}) + \prod_h(\delta\vartheta_h) \tag{D17}$$

and

$$\delta\vartheta_h = \vartheta_h - \frac{1}{4}\sum_{h'}(\delta\vartheta_{h'}). \tag{D18}$$

(The singular solutions are non-physical ones as they lead to vanishing wave-function.))

Another set of solutions is given by

$$\chi_1 = \frac{\gamma\kappa}{\pi} + \frac{\pi}{2},$$
$$\chi_2 = \frac{1}{2}\left(\sum_{h=1}^4 \theta_h - 2\chi_1\right) - \frac{\pi}{2}\,(\text{mod }\pi)\frac{\gamma}{2\pi}\left(\sum_{h=1}^4 \vartheta_h - 2\kappa\right)(\text{mod }\pi), \tag{D19}$$

with κ being one of the solutions of (D10). The label of these three solutions is $\{4, 1, 0\}$. Finally there is one solution labeled by $\{4, 2, 0\}$, for which

$$\chi_{1,2} = \frac{1}{4}\sum_h \theta_h \pm i\tanh^{-1}\left(1/\sqrt{3}\right) + \mathcal{O}(\gamma). \tag{D20}$$

The $\{4, 2, 2\}$, $\{4, 2, 1\}$ and $\{4, 2, 0\}$ solutions become degenerate in the SL as the equations determining their ϑ_h's become identical. For the same reason the $\{4, 1, 1\}$ and $\{4, 1, 0\}$ solutions form degenerate pairs having the same κ.

The general case

Now we attempt to classify the solutions of the equations (D1) in the $(n(\theta) \geq 2n(\chi))$ general case. It is clear, that in the scaling limit in each solution some of the χs (say $n(\kappa) \leq n(\chi)$ of them) will scale to $\pi/2$ in the form (D7) with the κs satisfying (D2) (with $n(\vartheta) = n(\theta)$). Now our task is to give the number of solutions at given $n(\theta)$, $n(\chi)$ and $n(\kappa)$, i.e. the number of solutions labeled by the same label $\{H, l, m\}$. Giving this number we have to make two points. First we suppose, that an equation (D1) has

$$N_{XXZ}(n(\theta), n(\chi)) = \binom{n(\theta)}{n(\chi)} \tag{D21}$$

nonequivalent solutions (one solution being a set of χs). This statement is strongly related to the question if the BA yields a complete set of eigenstates for the easy axis XXZ Heisenberg chain. Our other assumption is, that an equation of the type (D2) has

$$N_{XXX}(n(\vartheta), n(\kappa)) = \binom{n(\vartheta)}{n(\kappa)} - \binom{n(\vartheta)}{n(\kappa) - 1} \tag{D22}$$

different solutions. This is related to the question of completeness of the BA solutions for the isotropic case (in which a subset of the HLBAE are again of the type (D2). The difference of the two statements originates from the fact, that in the isotropic case the BA gives the $S^2 = S^z(S^z + 1)$ states only (i.e. not all of the states), while in the anisotropic case S^2 is not a good quantum number, and the BA should give all the $S^z \geq 0$ states. Now our conjecture is, that the number of those solution of (D1) in which $n(\kappa)(\leq n(\chi))$ of the χs scale to $\pi/2$ in the scaling limit is again (D22) (with $n(\theta) = n(\vartheta)$), and in each solution the $n(\kappa)$ of the χs which scale to $\pi/2$ are given by the different sets of κs solving (D2). This means, that if $n(\chi)(\geq n(\kappa))$ and a κ set solving (D2) are given, there is no further freedom, and there is only one possibility to complete the set of χs generated by the κs to a set solving (D1). In other words, in a class of solutions $\{H, l, m\}$ there are $N_{XXX}(H, \frac{1}{2}H - l)$ different states uniquely determined by the different solutions of (D2) independently of the value of m, (for which the only requirement is $l \geq m \geq 0$). This can be seen in the cases discussed, and is supported by the fact, that

$$\sum_{n(\kappa)=0}^{n(\chi)} N_{XXX}(n(\theta), n(\kappa)) = N_{XXZ}(n(\theta), n(\chi)). \tag{D23}$$

A consequence of this is that in the scaling limit the solutions can be collected into groups of degenerate states: at a given H and l the members of such a group are characterized by common ϑ_h and κ_j sets but differ in m what runs from 0 to l.

APPENDIX E:

In this appendix we show some solutions of the BAE of the finite lattice which has $S^z = 0$ and we give the symmetries of these states under $\hat{\Sigma}$. In these examples the length of the chain is always 20, the number of holes is 2, and we have one close pair. The equation (2.8) was solved by iteration which was started from the vicinity of the solutions of the HLBAE (2.17-2.18). After this the amplitudes (2.2) of some configurations (which we think dominate) were calculated and it was found that any of these amplitudes was ± 1 times the amplitude of the corresponding reflected configuration. (The relative error was 10^{-6} but it is plausible that the reflection symmetry holds exactly and the error comes only from the numerics.) The iteration converges fast also for much larger N, but the time needed for the calculation of the amplitudes grows rapidly with N resulting a strong upper limit for the number of lattice sites we can handle. For illustration, Table I shows some of the solutions we have found. There are two types of solutions: in type 1 the real part of the close pair v_c is between the two holes, while in type 2 it is located outside. It can be seen in Table I that in case of $I_0 = 1/2$ type 1 states are symmetric and type 2 ones are antisymmetric under $\hat{\Sigma}$ while the opposite is true for $I_0 = 0$. It should be noted that the connection between the type, I_0 and Σ is independent of the interval the rapidities are collected in because shifting one of the θs by π both the position of χ compared to the holes and the value of I_0 changes [6]. In our choice $(0 < v < \pi)$ type 1 resp. type 2 solutions scale to singlet (D5) resp. triplet (D6) states as it is indicated in Table I.

TABLES

I	$v,\ \theta$	$\gamma,\ v_c,\ I_0,\ \Sigma$	I	$v,\ \theta$	$\gamma,\ v_c,\ I_0,\ \Sigma$
-0.5	0.0462991		-1.5	0.1438185	
-1.5	0.1481864	$\gamma = 1$	-2.5	0.3092072	$\gamma = 1.5$
-2.5	**0.2663213**		**-3.5**	**0.5164954**	
-3.5	0.4241941	$v_c = 1.5046086 \pm i0.6433822$	-4.5	0.8408343	$v_c = 1.0670662 \pm i0.7483516$
-4.5	0.7049447	$(\to \text{singlet})$	**-5.5**	**1.6284527**	$(\to \text{singlet})$
-5.5	2.4271298		-6.5	2.2989503	
-6.5	**2.7097631**	$I_0 = \frac{1}{2}$	-7.5	2.6116381	$I_0 = 0$
-7.5	2.8692656		-8.5	2.8192199	
-8.5	2.9880692	$\Sigma = +1$	-9.5	2.9854038	$\Sigma = -1$
-9.5	3.0904861		-10.5	3.1355551	
-0.5	0.1361835		0.5	0.0407303	
-1.5	0.3022448	$\gamma = 1.5$	-0.5	0.1421697	$\gamma = 1$
-2.5	0.5124295		-1.5	0.2594036	
-3.5	0.8528326	$v_c = 0.8744719 \pm i0.7496363$	-2.5	0.4157572	$v_c = 1.3736073 \pm i0.6398024$
-4.5	1.6684991	$(\to \text{triplet})$	-3.5	0.6965562	$(\to \text{triplet})$
-5.5	**2.2976980**		-4.5	2.4190131	
-6.5	**2.6042345**	$I_0 = \frac{1}{2}$	**-5.5**	**2.7023676**	$I_0 = 0$
-7.5	2.8110406		-6.5	2.8629747	
-8.5	2.9773204	$\Sigma = -1$	-7.5	2.9825513	$\Sigma = +1$
-9.5	3.1276728		**-8.5**	**3.0850192**	

TABLE I. Some solutions of the BAE (2.8) of the $N = 20$ lattice with 8 real rapidities, 2 holes and one closed pair. In the first two columns the real rapidities v, the holes θ (emphasized by bold face) and the corresponding quantum numbers I can be found. The third column shows the parameter of the anisotropy γ, the rapidities of the closed pair v_c and the value of I_0 and Σ.

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